probability := chance of an event occurring

Variables

- discrete – described using counting numbers $\{0, 1, 2, \ldots\}$
- continuous – " " REAL numbers ($\mathbb{R}$)

→ ex. # of calls/hr
→ ex. height of adult female
wait time on hold

Probability for Discrete Variables – examples: probability as a function

- flipping a fair coin $\{H, T\}$ $P(H) = \frac{1}{2}$ $P(T) = \frac{1}{2}$
- rolling a fair die $\{1, 2, 3, 4, 5, 6\}$ $P(3) = \frac{1}{6}$ $P(odd) = \frac{3}{6} = \frac{1}{2}$
- pick 1 card from a standard deck ex. $P(A\heartsuit) = \frac{1}{52}$

$P(roll < 7) = \frac{6}{6} = 1$ ↑ 100%

$P(roll > 9) = \frac{0}{6} = 0 \rightarrow 0\%$

$$\underset{0\%}{0} \leq P(event) \leq \underset{100\%}{1}$$

Probability Distribution – table listing out all the possible options for the variable and the corresponding probabilities

Rules
- sum of all probabilities equals 1
- probabilities cannot be negative

ex. create a probability distribution for # of heads when 3 coins are flipped:

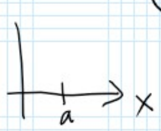
0: TTT (1)
1: HTT (2) THT (3) TTH (4)
2: HHT (5) HTH (6) THH (7)
3: HHH (8)

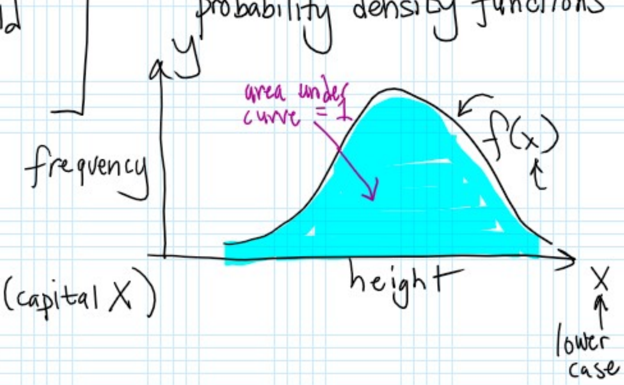
X	0	1	2	3
P(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$= \frac{8}{8} = 1$ ✓✓

Shifting from Discrete to Continuous Variables:
randomized continuous variables can be measured using functions → probability density functions

Continuous Probability Examples
- height of an adult female
- time spent waiting on hold
- life of a car battery





$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

random variable chosen (capital X)

then $P(a \leq X \leq b) = \int_a^b f(x)\,dx$

ex. given $f(x) = 0.006x(10 - x)$ for $0 \leq x \leq 10$ and $f(x) = 0$ for all other value of x

rewrite as a piecewise function:

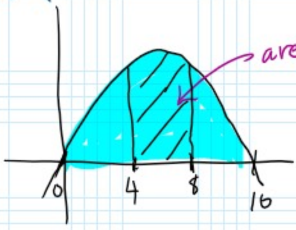
$$f(x) = \begin{cases} 0 & x < 0 \\ 0.006x(10-x) & 0 \leq x \leq 10 \leftarrow \\ 0 & x > 10 \end{cases}$$

a. verify that f(x) is a probability density function

$\quad\quad\quad 0 \quad\quad\quad\quad x > 10$

a. verify that $f(x)$ is a probability density function

$$\int_0^{10} (0.006x\,(10-x))\,dx = \int_0^{10} (0.06x - 0.006x^2)\,dx$$
$$= \left[\frac{6}{100}\cdot\frac{x^2}{2} - \frac{6}{1000}\cdot\frac{x^3}{3}\right]_0^{10}$$
$$= \frac{3}{\cancel{100}}[\cancel{100}] - \frac{2}{\cancel{1000}}\cdot\cancel{1000} = 3-2 = 1 \checkmark$$



b. find $P(4 \le X \le 8)$

$$\int_4^8 (0.06x - 0.006x^2)\,dx$$
$$= \frac{3}{100}\,x^2\Big|_4^8 - \frac{2}{1000}\,x^3\Big|_4^8$$
$$= \frac{3}{100}(64-16) - \frac{2}{1000}(512-64)$$
$$= \frac{3}{100}\cdot 48\cdot\boxed{\frac{10}{10}}_{\text{LCD}} - \frac{2}{1000}(448) = \frac{1440-896}{1000}$$
$$= \frac{544}{1000} = \boxed{\frac{68}{125}}$$
$\sim 54\%$

Do: Given $f(x) = xe^{-x}$ for $x \ge 0$; $f(x) = 0$ for $x < 0$

a. verify $f(x)$ is a probability density function

b. find $P(1 \le X \le 2) = \boxed{\frac{2}{e} - \frac{3}{e^2}}$

IBP $\quad u = x \quad du = dx \quad dv = e^{-x}\,dx \quad u = -x,\ -du = dx$
$$-e^u\,du$$
$$= -e^u = \boxed{-e^{-x} = v}$$

$$\int_0^\infty xe^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx$$
$$= \lim_{t\to\infty}\left[\underbrace{-xe^{-x}\Big|_0^t}_{uv} + \underbrace{\int_0^t e^{-x}\,dx}_{\int v\,du}\right]$$
$$= \lim_{t\to\infty}\left[-(te^{-t} - 0) \cancel{+} e^{-x}\Big|_0^t\right]$$
$$= \lim_{t\to\infty} -\frac{t}{e^t}\ \frac{\infty}{\infty} - (e^{-t} - e^0)$$
$\downarrow$ L'H
$$= -\lim_{t\to\infty}\frac{1}{e^t} - \lim_{t\to\infty} e^{-t}\ \boxed{+1} = 1$$
0

b. $P(1 \le X \le 2) = \int_1^2 xe^{-x}\,dx = -xe^{-x}\Big|_1^2 - e^{-x}\Big|_1^2$
$$= -(2e^{-2} - 1e^{-1}) - (e^{-2} - e^{-1})$$
$$= -\frac{2}{e^2} + \frac{1}{e} - \frac{1}{e^2} + \frac{1}{e}$$
$$= -\frac{3}{e^2} + \frac{2}{e}$$

$dv = e^{-x}\,dx$

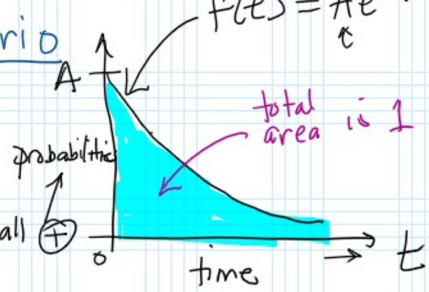
$v = \int e^{-x}\,dx \quad u = -x$
$\quad -du = dx$

$= -\int e^u\,du$

$-e^u \rightarrow -e^{-x}$

Decreased Exponential Scenario

- wait times
- equipment failure



$Ae^{-kt} \ge 0$ and $A\int_0^\infty e^{-kt}\,dt = 1$

what is A ?

$$A\lim_{R\to\infty}\int_0^R e^{-kt}\,dt = 1$$
$$-\frac{1}{k}A\lim_{R\to\infty} e^{-kt}\Big|_0^R = 1$$
$$-\frac{A}{k}\lim_{R\to\infty}\left(e^{-kR} - e^0\right) = 1$$
$$+\frac{A}{k}\ (+1) = 1 \Rightarrow \frac{A}{k} = 1 \quad \therefore A = k$$

$$\therefore f(t) = \begin{cases} 0 & t < 0 \\ ke^{-kt} & t \ge 0 \end{cases}$$

Mean – arithmetic average $\quad \frac{3+4+5}{3}$

Median – middle value (center of data)

Weighted Average

HW $\quad$ 20% → .2 $\quad$ 100

Weighted Average

Median – middle value (center of data)

HW	20%	→ .2	100
MT	30%	→ .3	60
FE	50%	→ .5	80

unweighted average $\frac{100+60+80}{3} = \frac{240}{3} = 80$

or $\frac{1}{3}(100) + \frac{1}{3}(60) + \frac{1}{3}(80)$

Weighted average: $100(.2) + 60(.3) + 80(.5)$
$= 20 + 18 + 40 = \boxed{78}$

"mu"

Generalize $\mu = \boxed{X_1 \cdot P(X_1)} + X_2 \cdot P(X_2) + \ldots + X_n P(X_n)$ ← finite # of values

mean

again, let's shift from a discrete scenario to a continuous one

$\mu = \lim_{n\to\infty} \sum_{i=1}^{n} x_i \cdot P(x_i) \Delta x \qquad P(x_i) = f(x)$

mean or average value $\boxed{\mu = \int_{-\infty}^{\infty} x \cdot f(x)\, dx}$

↳ for decreasing exponential:

$$\boxed{f(t) = \begin{cases} ke^{-kt} & t \geq 0 \\ 0 & t < 0 \end{cases}}$$

$$\mu = \int_0^{\infty} t\, k e^{-kt}\, dt$$ want to write k ITO μ

$\mu = k \lim_{A\to\infty} \int_0^A t e^{-kt}\, dt \qquad u = t \quad dv = e^{-kt}dt \qquad du = dt \quad v = -\frac{1}{k}e^{-kt}$

$\mu = k \lim_{A\to\infty} \left(-\frac{1}{k} t e^{-kt}\Big|_0^A + \frac{1}{k}\int_0^A e^{-kt}\, dt \right)$

$\mu = \cancel{k}\left(\cancel{\frac{1}{k}}\right) \lim_{A\to\infty} \left(-\left(\cancel{Ae^{-kt}}^{\,0} - 0\right) + -\frac{1}{k} e^{-kt}\Big|_0^A \right)$

$\mu = + \quad -\boxed{\frac{1}{k}} \lim_{A\to\infty} \left(\cancel{e^{-kA}}^{\,0} - \cancel{e^{0}}^{\,1} \right)$

$\mu = \frac{1}{k} \Rightarrow k = \frac{1}{\mu}$

$$\boxed{f(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{\mu} e^{-t/\mu} & t \geq 0 \end{cases}}$$

PROB. DENSITY FUNCTION FOR DECR. EXPONENTIAL

ex. Suppose a probability density function, $f(t)$, represents the amount of time a customer remains on hold to speak to a representative. Average wait time is 5 min.

a. find the probability that a random call is answered in the minute.

$P(0 \leq t \leq 1) = \frac{1}{5}\int_0^1 e^{-t/5}\, dt = \cancel{\frac{1}{5}(-5)}\, e^{-t/5}\Big|_0^1 = -\left(e^{-1/5} - 1\right)$

$\approx .1813$

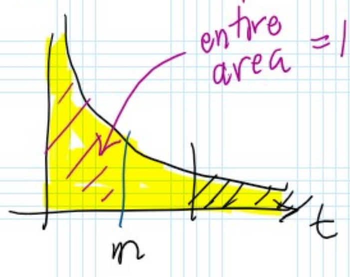
↳ 18.13%

b. find prob that caller waits more than 5 minutes

$P(t > 5) = \frac{1}{5}\int_5^{\infty} e^{-t/5}\, dt$

$= \frac{1}{5} \lim_{A\to\infty} \int_5^A e^{-t/5}\, dt$

$= -\cancel{5}\left(\cancel{\frac{1}{5}}\right) \lim_{A\to\infty} e^{-t/5}\Big|_5^A$



$= - \lim \left(\cancel{e^{-A/5}}^{\,0} - e^{-5/5} \right)$

$= -5\left(\frac{1}{5}\right) \lim_{A\to\infty} e \ldots$

$= - \lim_{A\to\infty} \left(\cancel{e^{-A/5}}^{\,0} \Big|_5 - e^{-5/5} \right)$

$= + e^{-1} = \frac{1}{e} \approx .0368$ $\boxed{3.68\%}$

c. determine the median wait time

↑ split area into 2 equal pieces

$\ln \frac{1}{2^1} = \ln 2^{-1}$

$\frac{1}{2} = \frac{1}{5} \int_m^\infty e^{-t/5}\, dt$

$\frac{5}{2} = \lim_{A\to\infty} \int_m^A e^{-t/5}\, dt$

$\frac{5}{2} = -5 \lim_{A\to\infty} e^{-t/5} \Big|_m^A$

$-\frac{1}{2} = \lim_{A\to\infty} \left(e^{-A/5} - e^{-m/5} \right)$

$\ln -\frac{1}{2} =$ $+ \cancel{\ln e}^{-m/5}$

$\ln 2^{-1}$

$+\ln 2 = + \frac{m}{5}$

$m = 5\ln 2 \approx 3.5$

median @ $t = 3.5$ min.

Work Section 6.6

work := amount of effort required to perform a task

requires force, F

Newton's Second Law of Motion : $F = ma$ (kg)

m is mass
a is acceleration

if constant acceleration, Work = force * distance $\boxed{W = F \cdot d}$ (m, N)

OR $\boxed{F = m \cdot \frac{d^2 s}{dt^2}}$ (k, $\frac{m}{s^2}$)

$s(t)$ or s position (s, m)

$s' = v$ velocity

$s'' = v' = a$ acceleration

unit is Newton-meter, 1 Joule

F units: $\underbrace{kg \cdot \frac{m}{s^2}}_{1 \text{ Newton}}$

if F is in lbs, d is in ft

$W = 1.36J$

ex. lift a 1.2kg (mass) book off the floor to put on 0.7m high desk.

(force exerted is offset by gravity)

$F = mg = (1.2)(9.8) = 11.76\,N$

↑ gravitational constant ~9.8

$W = Fd = (11.76)(0.7) \approx \boxed{8.2J}$

ex. 20 lb weight is lifted 6 ft off the ground

$W = Fd$

$= \underset{lb}{20}\ \underset{ft}{(6)} = \boxed{120 \text{ ft-lb}}$

if F is variable : $W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$

↑ $f(x)$

$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

$\boxed{W = \int_a^b f(x)\, dx}$ ←

ex. when a particle is located x feet from the origin,

ex. when a particle is located x feet from the origin, force acting on it is represented by $f(x) = x^2 + 2x$ ← variable force
How much work is done in moving particle from $x=1$ to $x=3$?

$$W = \int_1^3 (x^2+2x)\,dx = \left[\frac{x^3}{3} + x^2\right]_1^3$$
$$= \frac{1}{3}(26) + (8)$$
$$= \frac{26}{3} + \frac{24}{3} = \frac{50}{3} \text{ ft-lb} \quad 16\tfrac{2}{3}$$

Hooke's Law: $f(x) = kx$ ← spring constant
↑ force required to maintain the stretch on a spring stretched x units from its natural (at rest) position

ex. A force of 40N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15cm to 18 cm?

$N = kg \cdot \frac{m}{s^2}$

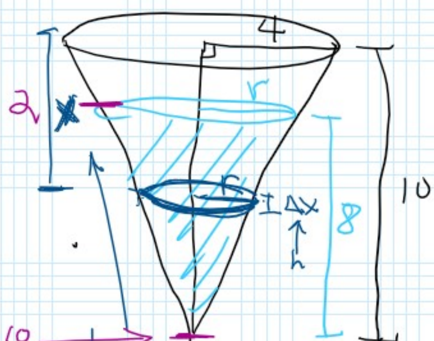
10 → 15 5 cm stretch $= .05$ m

$f(x) = kx$
$f(.05) = 40 = k(.05)$
$\Rightarrow k = 800$

$f(x) = 800x$

$$W = \int_{.05}^{.08} 800x\,dx = 1.56\text{J}$$

ex. A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m.
It is filled with water to a height of 8 m.
How much work is required to empty the tank by pumping all the water to the top of the tank?



distance each slice of water travels to be removed

bounds: $\int_2^{10}$

$V_{SLICE} = \pi r^2 \Delta x$
write r ITO x

$\frac{r}{h}$ $\frac{4}{10} = \frac{r}{10-x} \Rightarrow 10r = 4(10-x)$
$r = \frac{2}{5}(10-x)$

$= \pi\left(\frac{2}{5}(10-x)\right)^2 \Delta x$
$= \frac{4\pi}{25}(100 - 20x + x^2)\,\Delta x$

density of water $1000\ \frac{kg}{m^3}$

$V_{SLICE} = 160\pi\,(100 - 20x + x^2)\,\Delta x$

$W = F \cdot d$
$F \to m \cdot g$

$W = 9.8(160\pi)(100 - 20x + x^2)\,x \cdot \Delta x$
$= 1568\pi\,(100x - 20x^2 + x^3)\,\Delta x$
$= 1568\pi \int_2^{10} (100x - 20x^2 + x^3)\,dx$
$= 1568\pi \left(50x^2 - \frac{20}{3}x^3 + \frac{1}{4}x^4\right)\Big|_2^{10}$
$= 1568\pi\left(\frac{2048}{3}\right) \approx \boxed{3.4 \times 10^6 \text{ J}}$